Waiting times in a two-queue model with exhaustive and Bernoulli service

Weststrate, J.A.

TILBURG UNIVERSITY
KATHOLIEKE UNIVERSITEIT BRABANT
POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

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Jan A. Weststrate

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FACULTY OF ECONOMICS, TILBURG UNIVERSITY
P.O. BOX 90153, 5000 LE TILBURG, THE NETHERLANDS

ABSTRACT

THIS PAPER DEALS WITH THE CUSTOMERS' WAITING TIMES IN A TWO-QUEUE SYSTEM IN WHICH ONE QUEUE IS SERVED ACCORDING TO THE BERNOULLI SERVICE STRATEGY AND THE OTHER ONE ATTAINS EXHAUSTIVE SERVICE. EXACT RESULTS ARE DERIVED FOR THE LAPLACE-STIELTJES TRANSFORMS OF THE WAITING TIME DISTRIBUTIONS VIA AN ITERATION PROCEDURE. BASED ON THOSE RESULTS WE EXPRESS THE CUSTOMERS' MEAN WAITING TIMES IN THE SYSTEM PARAMETERS.

KEYWORDS:

POLLING SYSTEMS, BERNOULLI SERVICE, WAITING TIMES.

§ 1 Introduction and model description.

A system in which one server visits a set of queues, in some order, is commonly referred to as a polling system. A large number of queueing theoretic studies about polling systems has been published with the analysis focussing on characterizing the system performance. The vast majority of those studies considers polling systems with service policies commonly used in industry: the exhaustive, the gated and the limited service strategies.The main disadvantage of those traditional systems is the inability to exercise control and to affect their design by optimizing a performance measure such as the mean waiting time of an arbitrary customer in the system, the mean amount of work in the system or the mean cycle time.

As computer and telecommunication systems become more complicated and the processing power of micro processors becomes less expensive, the advantage of more sophisticated polling systems becomes apparent. Recently, more sophisticated service policies have been introduced. Among those are the fractional service policies, cf. Levy [1988a,b], and the Bernoulli service strategy, cf. Keilson and Servi [1986]. The present paper concerns a polling system in which one of the two queues has a Bernoulli service strategy.

This service policy is described as follows. When the server arrives at a queue he always serves one customer if the queue is not empty. If the queue is empty the server immediately starts to move to the next queue. After each service which does not leave the queue empty, the server serves another customer with probability 1-p and moves to the next queue with probability p. The advantage of this service policy is that the parameter p allows both flexible modelling and system optimization. From a theoretic point of view, another interesting property of the Bernoulli service policy is that it generalizes both the Exhaustive (a queue is served until it is empty) and the 1-Limited (when the queue is not empty, the server serves exactly one customer) service strategy.

Some recent studies concerning the Bernoulli service strategy are Keilson and Servi [1986], Servi [1986], Ramaswamy and Servi [1988] and Tedijanto [1989].

Our motivation is two-fold. Firstly we have a mathematical interest in the analysis of a generalisation of the most basic service disciplines, the exhaustive and the 1-limited service disciplines. Secondly, we would like to use the insight and exact results to be developed in the present study, for deriving and testing waiting time approximations in polling systems with Bernoulli service.

This paper concerns the customers' waiting times in a polling system with two queues in which one queue has a Bernoulli service policy with parameter $p\in[0,1]$ and the other one a Bernoulli service policy with a parameter equal to zero, the Exhaustive service policy. We shall indicate this system as the two queue Exhaustive/Bernoulli(p) system. For this system exact expressions for the Laplace-Stieltjes Transforms (LST) of the waiting time distributions are derived via an iteration procedure. So we need not solve a Riemann-Hilbert boundary value problem as in the 1-Limited/1-Limited case, Boxma and Groenendijk [1988], and most other two

queue cases cf. the discussion in Groenendijk [1990]:section 6.1. In section 2 we also show that the results for the two queue Exhaustive/Bernoulli(p) system generalize those of both the two queue Exhaustive/1-Limited system (p=1), studied by Groenendijk [1988], and the two queue Exhaustive/Exhaustive system (p=0) studied by for instance Takács [1968] and Eisenberg [1972]. In section 3 we express, based on the results in section 2, the customers' mean waiting times at both queues in the system parameters. As a check, it is shown that those expressions satisfy the pseudoconservation law for this system, cf. Boxma [1989]. Section 4 contains a summary and some plans for the future.
An appendix concludes this paper.

Model description.

A single server S serves two queues Q_1 and Q_2 in cyclic order. Both queues have an infinite buffer capacity. The arrival process at Q_i is a Poisson process with rate λ_i, $i\in\{1,2\}$. The service times at Q_i are independent, identically distributed stochastic variables with distribution $B_i(.)$, first moment β_i, second moment $\beta_i^{(2)}$ and Laplace-Stieltjes Transform $\tilde{B}_i(.)$. The utilization ρ_i at Q_i is defined by:

$$\rho_i := \lambda_i\beta_i, \quad i\in\{1,2\}. \quad (1.1)$$

The utilization of the server, ρ, is defined as:

$$\rho := \rho_1+\rho_2. \quad (1.2)$$

The service strategy at Q_i is Bernoulli(p_i), $p_i \in [0,1]$.

The successive switchover times from Q_i to $Q_{(i+1)\mathrm{mod}2}$ are independent, identically distributed stochastic variables, $\mathbf{s}_{i(i+1)\mathrm{mod}2}$ with distribution $S_{i(i+1)\mathrm{mod}2}(.)$, first moment $s_{i(i+1)\mathrm{mod}2}$, second moment $s^{(2)}_{i(i+1)\mathrm{mod}2}$ and Laplace-Stieltjes Transform $\tilde{S}_{i(i+1)\mathrm{mod}2}(.)$.

The first and second moment of the total switchover time during a cycle are denoted by, respectively:

$s := s_{12} + s_{21}$ (1.3)

and

$s^{(2)} := s^{(2)}_{12} + 2s_{12}s_{21} + s^{(2)}_{21}$. (1.4)

All stochastic processes are assumed to be mutually independent.

§ 2 Derivation of the generating functions of the queue lengths at polling instants.

In this section we determine the generating functions of the joint equilibrium queue length distributions at polling instants of Q_1 and Q_2. We proceed in two steps. In subsection 2.1 we derive recurrence relations between the generating functions. In subsection 2.2 those recurrence relations lead to explicit expressions for the generating functions of the steady-state queue lengths at polling instants.

§ 2.1 Determination of recurrence relations between the generating functions.

Let $\mathbf{x}_n^{(i)}$ denote the number of type-i customers in the system at the n-th polling instant of the server after t=0, n=1,2,..; i=1,2, and let $\mathbf{t}_n$ denote the queue which is visited during the n-th visit of the server after t=0.

The queue length process at Q_1 and Q_2 at successive polling epochs, $M:=\{(\mathbf{x}_n^{(1)},\mathbf{x}_n^{(2)}),\ n=1,2\ldots\}$, forms a vector Markov process. Note that this process is irreducible and aperiodic.

Define for $|z_1|\le 1, |z_2|\le 1$:

$$F_j^{(n)}(z_1,z_2) := E\{z_1^{\mathbf{x}_n^{(1)}}\ z_2^{\mathbf{x}_n^{(2)}}\ |\mathbf{t}_n = j\}, n=1,2,..;j=1,2. \quad (2.1.1)$$

A study of the transition probabilities of the Markov chain M yields recurrence relations for the generating functions of the queue lengths at polling instants, $F_1^{(n)}(z_1,z_2)$, $F_2^{(n)}(z_1,z_2)$, $|z_1|\le 1, |z_2|\le 1, n=1,2,\ldots$.

For the derivation of those relations we need some additional definitions and a theorem concerning the joint distribution of the length of a busy period and the number of customers at the end of that busy period in an M/G/1 queue with vacations and a Bernoulli service discipline. Define for such a queue:

$S_j(t,k)$:= the joint probability distribution of the length of a busy period and the queue length at the end of that busy period, conditioned on the fact that the busy period starts with j customers, $t\ge 0$; $k=0,1,\ldots$; $j=0,1,\ldots$.

Note that if $j=0$, i.e. there are no customers present when the server polls the Bernoulli queue, it is obvious that:

$$S_0(t,k):=0 \text{ if } k>0$$
$$S_0(t,0):=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } t\ge 0.\end{cases}$$

Also define the joint LST and generating function:

$$\sigma_j(\rho,r) := \sum_{k=0}^{\infty} r^k \int_{t=0}^{\infty} e^{-\rho t} d_t S_j(t,k), \ |r|\le 1;\ \mathrm{Re}\ \rho\ge 0;\ j=0,1,\ldots\ . \quad (2.1.2)$$

Using theorem I of Ramaswamy and Servi[1988] we can write for $|z_1|\le 1$; $|z_2|\le 1$; $p\in[0,1]$:

$$\sigma_j((1-z_1)\lambda_1, z_2) := \Omega_p(z_1,z_2)\ [\ z_2^j - \mu_2(z_1,p)^j\ ] + \mu_2(z_1,p)^j\ ,$$

$$j=0,1,\ldots\ ; \quad (2.1.3)$$

with

$$\Omega_p(z_1,z_2) := \frac{p\ \tilde{B}_2\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}}{z_2-(1-p)\tilde{B}_2\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}} \quad (2.1.4)$$

and for $|z_1|\le 1$, $\mu_2(z_1,p)$the unique solution of:

$$z_2 = (1-p)\tilde{B}_2\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\},\ |z_2|\le 1,\ p\in[0,1]. \quad (2.1.5)$$

Remark 2.1

Note that $\mu_2(z_1,p)$ is the joint LST and generating function of the length of a busy period and the number of customers served during that busy period of an ordinary M/G/1 queue with the same traffic characteristics as Q_2.

The existence of a unique root in (2.1.5) is demonstrated in Appendix 6 of Cohen[1982]. It is also shown there that if $\lambda_2\beta_2<1$:

$|\mu_2(z_1,p)| < 1$ for $p\in[0,1]$, $|z_1| \le 1$ except if $p=0$ and simultaneously $z_1=1$. In the latter case $\mu_2(z_1,p)=1$.

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We are now nearly ready to present the derivation of the recurrence relations between the generating functions (2.1.1). We have obtained those results by a tedious, but straightforward, calculation using indicator functions but we prefer to present them in another more intuitive way. Before we do that we have to introduce $\mu_1(z_2)$ as the unique solution of:

$z_1 = \tilde{B}_1\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}$, $|z_2|\le 1$ in the region $|z_1|\le 1$.

If the (n+1)-th polling epoch marks the beginning of a visit to Q_2 then:

1) because of the exhaustive service discipline at Q_1 the only type-1 customers present at Q_1, $\mathbf{x}^{(1)}_{n+1}$, are those who arrived during $\mathbf{s}_{12}$, the switch-over period between Q_1 and Q_2.

2) the type-2 customers at Q_2, $\mathbf{x}^{(2)}_{n+1}$, are composed of
   - the type-2 customers present at the n-th polling epoch, $\mathbf{x}^{(1)}_{n}$,
   - the type-2 customers who arrived during the subsequent visit of the server to Q_1, the n-th visit of the server after t=0,
   - the type-2 customers who arrived during $\mathbf{s}_{12}$.

Using those observations we can write for $|z_1|\le 1; |z_2|\le 1$:

$$E\{z_1^{\mathbf{x}^{(1)}_{n+1}} z_2^{\mathbf{x}^{(2)}_{n+1}} | \mathbf{t}_{n+1} = 2\} =$$

$$E\{z_1^{\#\text{type-1 arrivals during } \mathbf{S}_{12}} z_2^{\#\text{type-2 arrivals during } \mathbf{S}_{12}}\} \times$$

$$E\{z_2^{\#\text{type-2 arrivals during the n-th visit}} z_2^{\mathbf{x}^{(2)}_n} | \mathbf{t}_n=1\},$$

$$n=1,2,\ldots , \quad (2.1.6)$$

were we used the fact that the customers arrive according to Poisson processes.

For $|z_1|\le 1; |z_2|\le 1$:

$$E\{z_1^{\#\text{type-1 arrivals during } \mathbf{S}_{12}} z_2^{\#\text{type-2 arrivals during } \mathbf{S}_{12}}\} =$$

$$E\{e^{-(1-z_1)\lambda_1 \mathbf{s}_{12}} e^{-(1-z_2)\lambda_2 \mathbf{s}_{12}}\} = \tilde{S}_{12}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}. \quad (2.1.7)$$

If the server arrives at Q_1 and finds i type-1 customers present then we can view the visit period to Q_1 as a sequence of i independent identically distributed M/G/1 busy periods, c.f. Cohen[1982]:p.250. If we denote by $\mathbf{P}_k$ the k-th busy period in the sequence of i busy periods we can write for $|z_2|\le 1$:

$$E\{z_2^{\#\text{type-2 arrivals during the n-th visit}}\, z_2^{\mathbf{x}_n^{(2)}} \mid \mathbf{t}_n=1\}=$$

$$\sum_{i=0}^{\infty} E\{z_2^{\#\text{type-2 arrivals during } \mathbf{P}_1+\ldots+\mathbf{P}_i}\, z_2^{\mathbf{x}_n^{(2)}} (\mathbf{x}_n^{(1)}=i) \mid \mathbf{t}_n=1\}=$$

$$\sum_{i=0}^{\infty} E\{z_2^{\mathbf{x}_n^{(2)}} (\mathbf{x}_n^{(1)}=i) \mid \mathbf{t}_n=1\} E\{e^{-(1-z_2)\lambda_2 \mathbf{P}_1}\}^i =$$

$$\sum_{i=0}^{\infty} E\{z_2^{\mathbf{x}_n^{(2)}} (\mathbf{x}_n^{(1)}=i) \mid \mathbf{t}_n=1\} \mu_1(z_2)^i = E\{\mu_1(z_2)^{\mathbf{x}_n^{(1)}}\, z_2^{\mathbf{x}_n^{(2)}} \mid \mathbf{t}_n = 1\},$$

$$n=1,2,\ldots \quad (2.1.8)$$

Combining (2.1.6)..(2.1.8) and using definition (2.1.1) gives for $|z_1|\le 1;\ |z_2|\le 1$:

$$F_2^{(n+1)}(z_1,z_2)= \tilde{S}_{12}\{(1-z_1)\lambda_1 + (1-z_2)\lambda_2\}\ F_1^{(n)}(\mu_1(z_2),z_2),$$

$$n=1,2,\ldots \quad (2.1.9)$$

If the (n+1)-th polling epoch marks the beginning of a visit to Q_1 then:

1) the type-1 customers at Q_1, $\mathbf{x}_{n+1}^{(1)}$, are composed of:
   -the type-1 customers present at the n-th polling instant, $\mathbf{x}_n^{(1)}$,
   -the type-1 arrivals during the visit of the server to Q_2, the n-th visit,

-the type-1 arrivals during $\mathbf{s}_{21}$, the switchover period between Q_2 and Q_1.

2) the type-2 customers at Q_2 are composed of:
-the type-2 customers present at the end of the previous visit, which we shall denote by $\mathbf{u}_n^{(2)}$,
-the type-2 arrivals during $\mathbf{s}_{21}$.

Using those observations we can write for $|z_1|\le 1; |z_2|\le 1$:

$$E\{z_1^{\mathbf{x}_{n+1}^{(1)}} z_2^{\mathbf{x}_{n+1}^{(2)}} | \mathbf{t}_{n+1}=1\} =$$

$$E\{z_1^{\text{type-1 arrivals during } \mathbf{s}_{21}} z_2^{\text{type-2 arrivals during } \mathbf{s}_{21}}\}.$$

$$E\{z_1^{\text{type-1 arrivals during the n-th visit}} z_1^{\mathbf{x}_n^{(1)}} z_2^{\mathbf{u}_n^{(2)}} \mid \mathbf{t}_n = 2\},$$

$$n=1,2,\ldots \quad (2.1.10)$$

were we used the fact that the customers arrive according to Poisson processes.

For $|z_1|\le 1; |z_2|\le 1$:

$$E\{z_1^{\text{type-1 arrivals during } \mathbf{s}_{21}} z_2^{\text{type-2 arrivals during } \mathbf{s}_{21}}\} =$$

$$\tilde{S}_{21}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\} \quad . \quad (2.1.11)$$

Note that for $|z_1|\le 1; |z_2|\le 1$:

$\sigma_j((1-z_1)\lambda_1, z_2)$=the joint generating function of the number of type-1 arrivals during a busy period of the Bernoulli queue and the number of type-2 customers in the system at the end of that busy period conditioned on the fact that the busy period starts with j (type-2) customers, $j=0,1,\dots$.

Using this fact and formula (2.1.3) we can write for $|z_1|\le 1; |z_2|\le 1$:

$$E\{z_1^{\text{type-1 arrivals during the n-th visit}}\, z_1^{\mathbf{x}_n^{(1)}}\, z_2^{\mathbf{u}_n^{(2)}} \,|\, \mathbf{t}_n=2\}=$$

$$\sum_{i,j=0}^{\infty} z_1^i\, \sigma_j((1-z_1)\lambda_1, z_2)\ \Pr\{\ \mathbf{x}_n^{(1)}= i,\ \mathbf{x}_n^{(2)}= j|\mathbf{t}_n=2\} =$$

$$\Omega_p(z_1,z_2) \sum_{i,j=0}^{\infty} z_1^i\ [z_2^j - \mu_2(z_1,p)^j]\ \Pr\{\ \mathbf{x}_n^{(1)}= i,\ \mathbf{x}_n^{(2)}=j|\mathbf{t}_n=2\} +$$

$$\sum_{i,j=0}^{\infty} z_1^i\ \mu_2(z_1,p)^j\ \Pr\{\ \mathbf{x}_n^{(1)}= i,\ \mathbf{x}_n^{(2)}= j|\mathbf{t}_n=2\} =$$

$$\Omega_p(z_1,z_2)[F_2^{(n)}(z_1,z_2)- F_2^{(n)}(z_1,\mu_2(z_1,p))] + F_2^{(n)}(z_1,\mu_2(z_1,p)),$$

$$n=1,2,\dots \qquad (2.1.12)$$

Combining(2.1.10),...,(2.1.12) gives for $|z_1|\le 1; |z_2|\le 1$:

$$F_1^{(n+1)}(z_1,z_2) = \tilde{S}_{21}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}F_2^{(n)}(z_1,\mu_2(z_1,p)) +$$

$$\tilde{S}_{21}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}\Omega_p(z_1,z_2)[F_2^{(n)}(z_1,z_2)-F_2^{(n)}(z_1,\mu_2(z_1,p))],$$

$$n=0,1,\dots \qquad (2.1.13)$$

Assumption 2.1

From now on the vector Markov process $M=\{(\mathbf{x}_n^{(1)},\mathbf{x}_n^{(2)}),\ n=1,2,..\}$, is assumed to be positive recurrent.

Remark 2.2

It is easily seen that $\rho<1$ and $\dfrac{\lambda_2 p\ s}{1-\rho}<1$ are necessary conditions for positive recurrence of the Markov process M. This leads to the condition:

$\lambda_2 ps + \rho < 1$; we believe this condition is also sufficient but we have not formally proved the sufficiency.

■

From assumption 2.1 and the irreducibility and aperiodicity of the process M it follows that the vector Markov process M has a limiting distribution equal to its stationary distribution.

Define for $|z_1|\le 1; |z_2|\le 1$:

$$F_j(z_1,z_2) := \lim_{n\to\infty} F_j^{(n)}(z_1,z_2),\ j=1,2. \quad (2.1.14)$$

Using the recurrence relations (2.1.9) and (2.1.13) together with definition (2.1.14) we can relate $F_1(z_1,z_2)$, $F_2(z_1,z_2)$ in the following way:

for $|z_1|\le 1; |z_2|\le 1$:

$$F_1(z_1,z_2) = \tilde{S}_{21}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}\Omega_p(z_1,z_2)F_2(z_1,z_2) +$$

$$\tilde{S}_{21}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}[1-\Omega_p(z_1,z_2)]F_2(z_1,\mu_2(z_1,p)), \quad (2.1.15)$$

$$F_2(z_1,z_2) = \tilde{S}_{12}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}F_1(\mu_1(z_2),z_2). \quad (2.1.16)$$

§ 2.2 <u>Determination of explicit expressions for</u> $F_1(z_1,z_2)$ <u>and</u> $F_2(z_1,z_2)$.

If $\tilde{X}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}$ is the LST with argument $(1-z_1)\lambda_1+(1-z_2)\lambda_2$ of a certain stochastic variable we define for notational convenience:

$\tilde{X}\{z_1,z_2\}:=\tilde{X}\{(1-z_1)\lambda_1+(1-z_2)\lambda_2\}$ and

$\tilde{X}\tilde{Y}\{z_1,z_2\}:=\tilde{X}\{z_1,z_2\}\tilde{Y}\{z_1,z_2\}$, $|z_1|\le 1; |z_2|\le 1$.

Taking $z_1=\mu_1(z_2)$ in relation (2.1.15) and combining the result with (2.1.16) gives for $|z_1|\le 1; |z_2|\le 1$ and $p\in[0,1]$:

$$F_2(z_1,z_2) = \tilde{S}_{12}\{z_1,z_2\}\tilde{S}_{21}\{\mu_1(z_2),z_2\}\Omega_p(\mu_1(z_2),z_2)F_2(\mu_1(z_2),z_2) +$$

$$\tilde{S}_{12}\{z_1,z_2\}\tilde{S}_{21}\{\mu_1(z_2),z_2\}[1-\Omega_p(\mu_1(z_2),z_2)]F_2(\mu_1(z_2),\mu_2(\mu_1(z_2),p)).$$

(2.2.2)

Before we derive an iteration relation we need some additional definitions.

Define for $|z|\le 1$ and $p\in[0,1]$:

$\delta_p^{(0)}(z):=z$,

$\delta_p(z)=\delta_p^{(1)}(z):=\mu_2(\mu_1(z),p)$,

$$\delta_p^{(n)}(z):=\delta_p^{(1)}(\delta_p^{(n-1)}(z)),\ n=1,2,\ldots\quad (2.2.3)$$

Remark 2.3(Interpretation of $\delta_p^{(1)}(z)$)

Define :

$\mathbf{n}$:= the number of customers served during a busy period of an ordinary M/G/1 queue,

$\mathbf{v}^2(\mathbf{P}^{(1)})$:=the number of customers that arrive at Q_2 during a busy period of Q_1,

$\mathbf{v}^1(\mathbf{P}^{(2)}(\mathbf{n}))$:=the number of customers that arrive at Q_1 during a busy period at an ordinary M/G/1 queue with the same traffic characteristics as Q_2, at which busy period $\mathbf{n}$ customers are being served.

Using those definitions we can write for $|z|\le 1$ and $p\in[0,1]$:

$$\delta_p^{(1)}(z))=E\{(1-p)^{\mathbf{n}}\, z^{\mathbf{v}^2(\mathbf{P}^{(1)})_1+\ldots+\mathbf{v}^2(\mathbf{P}^{(1)})_{\mathbf{v}^1(\mathbf{P}^{(2)}(\mathbf{n}))}}\}.$$

Note that $\mathbf{v}^2(\mathbf{P}^{(1)})_1+\ldots+\mathbf{v}^2(\mathbf{P}^{(1)})_{\mathbf{v}^1(\mathbf{P}^{(2)}(\mathbf{n}))}$ denotes the number of arrivals at Q_2 during a sequence of $\mathbf{v}^1(\mathbf{P}^{(2)}(\mathbf{n}))$ busy periods at Q_1.

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Taking $z_1=\mu_1(z_2)$ in (2.2.2), reordering terms in the resulting equation and then replacing z_2 by $\delta_p(z)$ gives for $|z|\le 1$ and $p\in[0,1]$

$$F_2(\mu_1(\delta_p(z)),\delta_p(z))=$$

$$\frac{\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(\delta_p(z)),\delta_p(z)\}[1-\Omega_p(\mu_1(\delta_p(z)),\delta_p(z))]}{1-\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(\delta_p(z)),\delta_p(z)\}\Omega_p(\mu_1(\delta_p(z),\delta_p(z))}\ F_2(\mu_1(\delta_p(z)),\delta_p^{(2)}(z)).\quad (2.2.4)$$

Taking $z_2=\mu_2(z_1)$ in (2.2.2) and replacing z_1 by $\mu_1(z)$ in the resulting equation gives for $|z|\le 1$ and $p\in[0,1]$:

$$F_2(\mu_1(z),\delta_p(z))= \quad \tilde{S}_{21}\{\mu_1(z),\delta_p(z)\}\tilde{S}_{21}\{\mu_1(\delta_p(z)),\delta_p(z)\}.$$

$$\Big[\Omega_p(\mu_1(\delta_p(z)),\delta_p(z))F_2(\mu_1(\delta_p(z),\delta_p(z))+$$

$$[1-\Omega_p(\mu_1(\delta_p(z)),\delta_p(z))]F_2(\mu_1(\delta_p(z)),\delta_p^{(2)}(z))]\Big]. \tag{2.2.5}$$

Combining (2.2.4) and (2.2.5) leads for $|z|\le 1$ and $p\in[0,1]$ to:

$$F_2(\mu_1(z),\delta_p(z))=D_p(\delta_p(z))\ F_2(\mu_1(\delta_p(z)),\delta_p^{(2)}(z)), \tag{2.2.6}$$

with

$$D_p(\delta_p(z))= \frac{\tilde{S}_{12}\{\mu_1(z),\delta_p(z)\}\tilde{S}_{21}\{\mu_1(\delta_p(z)),\delta_p(z)\}[1-\Omega_p(\mu_1(\delta_p(z)),\delta_p(z))]}{1-\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(\delta_p(z)),\delta_p(z)\}\Omega_p(\mu_1(\delta_p(z),\delta_p(z))}. \tag{2.2.7}$$

Replacement of z by $\delta_p(z)$ in (2.2.6) gives for $|z|\le 1$ and $p\in[0,1]$:

$$F_2(\mu_1(\delta_p(z)),\delta_p^{(2)}(z))=D_p(\delta_p^{(2)}(z))F_2(\mu_1(\delta_p^{(2)}(z)),\delta_p^{\ (3)}(z)). \tag{2.2.8}$$

Substitution of (2.2.8) in (2.2.6) gives for $|z|\le 1$ and $p\in[0,1]$:

$$F_2(\mu_1(z),\delta_p(z))=D_p(\delta_p^{(1)}(z))D_p(\delta_p^{(2)}(z))F_2(\mu_1(\delta_p^{(2)}(z)),\delta_p^{(3)}(z)). \tag{2.2.9}$$

Repeating this procedure n times yields for $|z|\le 1$ and $p\in[0,1]$:

$$F_2(\mu_1(z),\delta_p(z))=[\prod_{k=1}^{n} D_p(\delta_p^{(k)}(z))]\ F_2(\mu_1(\delta_p^{(n)}(z)),\delta_p^{(n+1)}(z)). \tag{2.2.10}$$

In the appendix we show that for $|z| \le 1$ and $p \in [0,1]$:

1) $\lim_{n\to\infty} \delta_p^{(n)}(z) = a$ for some $a \in (0,1]$,

2) $\prod_{k=1}^{\infty} D_p(\delta_p^{(k)}(z)) = b$ for some $b \in (0,\infty)$.

Then, using the continuity of $F_2(u,v)$ in u and v,

$$\lim_{n\to\infty} F_2(\mu_1(\delta_p^{(n)}(z)), \delta_p^{(n+1)}(z)) = F_2(\mu_1(a), a) =: C^*. \tag{2.2.11}$$

Because of relation (2.2.10) we can write for $|z| \le 1$ and $p \in [0,1]$:

$$F_2(\mu_1(z), \delta_p(z)) = C^* \prod_{k=1}^{\infty} D_p(\delta_p^{(k)}(z)). \tag{2.2.12}$$

Taking $z_1 = \mu_1(z)$ and $z_2 = z$ in (2.2.2) and reordering terms gives for $|z| \le 1$ and $p \in [0,1]$:

$$F_2(\mu_1(z), z) = \frac{\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z), z\}[1 - \Omega_p(\mu_1(z), z)]}{1 - \tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z), z\}\Omega_p(\mu_1(z), z)} F_2(\mu_1(z), \delta_p(z)). \tag{2.2.13}$$

Combining (2.2.12) and (2.2.13) gives for $|z| \le 1$ and $p \in [0,1]$:

$$F_2(\mu_1(z), z) = \frac{\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z), z\}[1 - \Omega_p(\mu_1(z), z)]}{1 - \tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z), z\}\Omega_p(\mu_1(z), z)} C^* \prod_{k=1}^{\infty} D_p(\delta_p^{(k)}(z)). \tag{2.2.14}$$

From (2.1.16) it follows that for $|z| \le 1$:

$$F_2(\mu_1(z), z) = \tilde{S}_{12}\{\mu_1(z), z\} F_1(\mu_1(z), z). \tag{2.2.15}$$

Combining (2.2.14) and (2.2.15) and substituting the resulting expression for $F_1(\mu_1(z),z)$ in (2.1.16) yields for $|z_1|\le 1; |z_2|\le 1$ and $p\in[0,1]$:

$$F_2(z_1,z_2)=\frac{\tilde{S}_{12}\{z_1,z_2\}\ \tilde{S}_{21}\{\mu_1(z_2),z_2\}[1-\Omega_p(\mu_1(z_2),z_2)]}{1-\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z_2),z_2\}\Omega_p(\mu_1(z_2),z_2)}\ C^*\prod_{k=1}^{\infty}D_p(\delta_p^{(k)}(z_2)). \tag{2.2.16}$$

Using now the fact that $F_2(z_1,z_2)$ is the generating function of a probability distribution, so that $F_2(1,1)=1$, applying L'Hôpital's rule to the right hand side of equation (2.2.16) and using:

1) $\left[\dfrac{d}{dz}\ \mu_1(z)\right]_{z=1} = \dfrac{\lambda_2\ \beta_1}{1-\rho_1}$,

2) $\left[\dfrac{d}{dz}\ \tilde{B}_2\{\mu_1(z),z\}\right]_{z=1} = \dfrac{\rho_2}{1-\rho_1}$,

3) $\left[\dfrac{d}{dz}\ \tilde{S}_{ij}\{\mu_1(z),z\}\right]_{z=1} = \dfrac{s_{ij}\ \lambda_2}{1-\rho_1}$, $i,j\in\{1,2\}$,

we find that:

$$C^* = [1-p\ \frac{\lambda_2\ s}{1-\rho}\ ][\prod_{k=1}^{\infty}D_p(\delta_p^{(k)}(1))]^{-1}. \tag{2.2.17}$$

Define for $|z|\le 1$ and $p\in[0,1]$:

$$G_p(z):=\frac{\tilde{S}_{21}\{\mu_1(z),z\}[1-\Omega_p(\mu_1(z),z)]}{1-\tilde{S}_{12}\tilde{S}_{21}\{\mu_1(z),z\}\Omega_p(\mu_1(z),z)}\ . \tag{2.2.18}$$

Using (2.2.17), (2.2.18) and (2.1.15) and (2.1.16) we get explicit expressions for the generating functions of the queue lengths at polling instants, for $|z_1|\le 1; |z_2|\le 1$ and $p\in[0,1]$:

$$F_1(z_1,z_2)=[1 - p\ \frac{\lambda_2 s}{1-\rho}]\tilde{S}_{21}\{z_1,z_2\}.$$

$$\Big[\ \Omega_p(z_1,z_2)\tilde{S}_{12}\{z_1,z_2\}G_p(z_2)\prod_{k=1}^{\infty}\{D_p(\delta_p^{(k)}(z_2))/D_p(\delta_p^{(k)}(1))\}\ +$$

$$[1-\ \Omega_p(z_1,z_2)]\tilde{S}_{12}\{z_1,\mu_2(z_1)\}G_p(\mu_2(z_1))\prod_{k=1}^{\infty}\{D_p(\delta_p^{(k)}(\mu_2(z_1)))/D_p(\delta_p^{(k)}(1))\}\Big],$$

(2.2.19)

$$F_2(z_1,z_2)=[1 - p\ \frac{\lambda_2 s}{1-\rho}]\tilde{S}_{12}\{z_1,z_2\}G_p(z_2)\prod_{k=1}^{\infty}\{D_p(\delta_p^{(k)}(z_2))/D_p(\delta_p^{(k)}(1))\}.$$

(2.2.20)

<u>Remark</u> <u>2.4</u> (Comparison with the two-queue Exhaustive/1-Limited model)

If we take the Bernoulli parameter, p, equal to one, the model under consideration in this paper becomes a two queue Exhaustive/1-Limited model (cf. Groenendijk [1990]:§6.3).

For p=1 we get the following expressions:

1) $\mu_2(z,1)=0,\ |z|\le 1;$

2) $\delta_1^{(k)}(z)=0,\ |z|\le 1,\ k=1,2,\dots\ ;$

3) $$D_1(\delta_1^{(k)}(z))=\frac{\tilde{S}_{12}\{\mu_1(z),0\}}{\tilde{S}_{12}\{\mu_1(0),0)},\quad |z|\le 1\ ,\ k=1;$$

$$= 1\qquad ,\ |z|\le 1,\ k=2,3,\dots\ ;$$

4) $$G_1(z)=\frac{\tilde{S}_{21}\{\mu_1(z),z\}[z-\tilde{B}_2\{\mu_1(z),z\}]}{z-\ \tilde{S}_{12}\tilde{S}_{21}\tilde{B}_2\{\mu_1(z),z\}},\quad |z|\le 1\ .$$

Using these relations (2.2.20) becomes for p=1 and $|z_1|\le 1;|z_2|\le 1$:

$$F_2(z_1,z_2)\Big|_{p=1} = [1 - \frac{\lambda_2 s}{1-\rho}]\tilde{S}_{12}\{z_1,z_2\} \cdot$$

$$\frac{\tilde{S}_{21}\{\mu_1(z),z\}[z-\tilde{B}_2\{\mu_1(z),z\}]}{z - \tilde{S}_{12}\tilde{S}_{21}\tilde{B}_2\{\mu(z),z\}} \; \frac{\tilde{S}_{12}\{\mu_1(z),0\}}{\tilde{S}_{12}\{1,0\}}, \tag{2.2.21}$$

Equation (2.2.21) is the same as (6.68) in Groenendijk [1990].
From (2.2.21) it follows that :

$$F_2(z,\mu_2(z,1))\Big|_{p=1} = F_2(z,0)\Big|_{p=1} = [1 - \frac{\lambda_2 s}{1-\rho}] \frac{\tilde{S}_{12}\{z,0\}}{\tilde{S}_{12}\{1,0\}}. \tag{2.2.22}$$

We can now write (2.1.15) in terms of (2.2.21) and (2.2.22):

$$F_1(z_1,z_2)\Big|_{p=1} = \frac{\tilde{B}_2\ \tilde{S}_{21}\{z_1,z_2\}}{z_2} F_2(z_1,z_2)\Big|_{p=1} +$$

$$\frac{S_{21}\{z_1,z_2\}[z_2-B_2\{z_1,z_2\}]}{z_2} F_2(z_1,0)\Big|_{p=1}. \tag{2.2.23}$$

<u>Remark</u> <u>2.5</u> (Comparison with the two-queue Exhaustive/Exhaustive model)
One of the earliest two-queue models studied was the so-called alternating priority model (with *zero* switchover times). In this model the server does not leave a queue until it is emptied (nowadays it is called exhaustive service).
For the analysis of this model see for instance Avi-Itzhak et al. [1965] and Takács [1968]. Later on Eisenberg [1971] and Sykes [1970] generalized this model to a two-queue alternating priority model *with* switchover times. In all those studies the server was assumed to be idle if the system was empty. Eisenberg [1972] generalized the two-queue alternating priority model to an M-queue model with non-zero switchover

times. He also assumed that the server keeps on switching if the system is empty.

If we take M=2 we can view this model as a special case of the two-queue Exhaustive/Bernoulli model studied in this paper. We shall show that if we take the Bernoulli parameter equal to zero we get the same expressions for the generating functions of the queue lengths at polling instants as Eisenberg [1972].

For p=0 we get the following expressions:

1) $\delta_0^{(k)}(1)=1,\ |z|\le 1,\ k=0,1,\ldots$;

2) $\Omega_0(z_1,z_2) = 0,\ |z_1|\le 1; |z_2|\le 1;$

3) $D_0(\delta_0^{(k)}(z))=\tilde{S}_{12}\{\mu_1(\delta_0^{(k-1)}(z)),\delta_0^{(k)}(z)\}\tilde{S}_{21}\{\mu_1(\delta_0^{(k)}(z)),\delta_0^{(k)}(z)\},$
$|z|\le 1\ ,\ k=1,2,\ldots$;

4) $G_0(z)= \tilde{S}_{21}\{\mu_1(z),z\},\ |z|\le 1.$

Let us now define for $|z|\le 1$:

$\varepsilon^{(0)}(z):=z,$

$\varepsilon\ (z)=\varepsilon^{(1)}(z):=\mu_1(\mu_2(z,0)),$

$\varepsilon^{(n)}(z):=\varepsilon^{(1)}(\varepsilon^{(n-1)}(z)),\ n=1,2,\ldots$.

Using this definition and the above expressions we get for p=0 and $|z_1|\le 1, |z_2|\le 1$:

$$F_1(z_1,z_2)\Big|_{p=0} = \tilde{S}_{21}\{z_1,z_2\}\prod_{n=1}^{\infty}\tilde{S}_{21}\{\varepsilon^{(n)}(z_1),\mu_2(\varepsilon^{(n-1)}(z_1))\}\cdot \prod_{n=0}^{\infty}\tilde{S}_{12}\{\varepsilon^{(n)}(z_1),\mu_2(\varepsilon^{(n)}(z_1))\}, \tag{2.2.24}$$

$$F_2(z_1,z_2)\Big|_{p=0} = \tilde{S}_{12}\{z_1,z_2\}\prod_{n=1}^{\infty}\tilde{S}_{12}\{\mu_1(\delta_0^{(n-1)}(z_2)),\delta_0^{(n)}(z_2)\}\cdot \prod_{n=0}^{\infty}\tilde{S}_{21}\{\mu_1(\delta_0^{(n)}(z_2)),\delta_0^{(n)}(z_2)\}. \tag{2.2.25}$$

If we combine relations (18) and (31) of Eisenberg [1972] we get the same expressions for the generating functions of the queue lengths at polling instants as (2.2.24) and (2.2.25).

■

§ 3 The waiting times.

This section is concerned with the customers' waiting times at the queues. In subsection 3.1 we deal with Q_1, the queue with an exhaustive service discipline, and in subsection 3.2 we consider Q_2, the Bernoulli queue. In each case we first give the LST of the waiting time distribution at that particular queue expressed in the generating functions derived in the previous section, and subsequently we calculate the mean waiting time.

Define for $i\in\{1,2\}$:

$\mathbf{w}_i$:= the waiting time of a type-i customer,

$W_i(t):=\Pr\{\mathbf{w}_i<t\}$, $t\geq 0$,

$\tilde{W}_i(\rho):=\int_{t=0}^{\infty} e^{-\rho t}d_tW_i(t)$, Re $\rho\geq 0$.

§ 3.1 The waiting time at the exhaustive queue.

The generating function of the queue lengths at polling instants of Q_1, the queue with the exhaustive service strategy, and the waiting time, $\mathbf{w}_1$, of a type-1 customer are related as follows (cf. Watson [1985]:p.526):

$$E\{e^{-(1-z)\lambda_1\mathbf{w}_1}\}=\frac{1-\lambda_1\beta_1}{\left[\frac{d}{dz}F_1(z,1)\right]_{z=1}}\ \frac{1-F_1(z,1)}{\tilde{B}_1\{z,1\}-z},\ |z|\leq 1. \tag{3.1.1}$$

Taking the derivative of (3.1.1) and evaluating it in z=1 yields:

$$\lambda_1 E\{w_1\} = \frac{d}{dz} \left. \frac{1-F_1(z,1)}{1-z} \right|_{z=1} \frac{1}{F_1'(y,1)|_{y=1}} + \frac{\lambda_1^2 \beta_1^{(2)}}{2(1-\rho_1)}, \tag{3.1.2}$$

with $F_1'(y,1) := \frac{d}{dy} F_1(y,1)$.

To get an expression for the mean waiting time at Q_1 we first expand, using equation (2.2.19), $\frac{1-F_1(z,1)}{1-z}$ in a power series in the neighborhood of $z=1$. Noting that for $p \in [0,1]$ (cf. (2.2.20)):

$$F_2(1,\mu_2(1,p)) = 1 - p \frac{\lambda_2 s}{1-\rho}, \tag{3.1.3}$$

we find after a lengthy but straightforward calculation the following expression for the mean waiting time at Q_1:

$$E\{w_1\} = \frac{\lambda_1 \beta_1^{(2)} + \lambda_2 \beta_2^{(2)}}{2(1-\rho_1)} + \frac{(1-\rho)}{(1-\rho_1)} \frac{s^{(2)}}{2s} +$$

$$\frac{(1-\rho)}{(1-\rho_1)} \left[\frac{1-p}{p} \frac{\lambda_2 \beta_2^2}{1-\rho} + \frac{\beta_2}{p} - \left(1 - \frac{p\lambda_2 s}{(1-\rho)}\right) \frac{s_{21}}{s} \frac{\beta_2}{p} \right] -$$

$$\frac{(1-\rho)}{(1-\rho_1)} \left(1 - \frac{p\lambda_2 s}{(1-\rho)}\right) \frac{\beta_2}{\lambda_1 p s} \left[\frac{d}{dz} H_1(\mu_2(z)) \right]_{z=1}, \tag{3.1.4}$$

$$\text{with } H_1(\mu_2(z)) := \prod_{k=0}^{\infty} \{ D_p(\delta_p^{(k)}(\mu_2(z_1))) / D_p(\delta_p^{(k)}(1)) \}. \tag{3.1.5}$$

§ 3.2 The mean waiting time at the Bernoulli queue.

Using theorem 3.1 of Tedijanto [1989] we can relate the generating function of the queue lengths at polling instants of Q_2, the queue with the Bernoulli service strategy, and the LST of the distribution of the waiting time at Q_2, $\mathbf{w}_2$, in the following way:

$$E\{e^{-(1-z)\lambda_2\mathbf{w}_2}\}=\frac{p}{1-F_2(1,\mu_2(1,p))}\,\frac{F_2(1,z)-F_2(1,\mu_2(1,p))}{z-(1-p)\tilde{B}_2\{1,z\}},\qquad p\in[0,1],\ |z|\le1. \tag{3.2.1}$$

Using relation (3.1.3) we can write (3.2.1) as follows:

$$E\{e^{-(1-z)\lambda_2\mathbf{w}_2}\}=\frac{1-\rho}{\lambda_2 s}\,\frac{F_2(1,z)-1}{z-(1-p)\tilde{B}_2\{1,z\}}+\frac{p}{z-(1-p)\tilde{B}_2\{1,z\}},$$
$$p\in[0,1];\ |z|\le1. \tag{3.2.2}$$

Taking the derivative of (3.2.2) and evaluating it in z=1 gives:

$$\lambda_2E\{\mathbf{w}_2\}=\frac{1-\rho}{\lambda_2 s}\,\frac{F_2'(1,z)\big|_{z=1}}{p}-\frac{1-(1-p)\rho_2}{p},\quad p\in[0,1]. \tag{3.2.3}$$

To express $E\{\mathbf{w}_2\}$ in the system parameters we expand $F_2(1,z)$ (cf.(2.2.20)) in a power series in the neighborhood of z=1. After some further calculations we get:

$$E\{\mathbf{w}_2\}=\frac{\lambda_1\beta_1^{(2)}+\lambda_2\beta_2^{(2)}}{2(1-\rho_1)(1-\rho)}\,\frac{1}{1-\frac{p\lambda_2 s}{(1-\rho)}}+\frac{1}{(1-\rho_1)}\left[\frac{s^{(2)}}{2s}+\beta_2\right]\frac{1}{1-\frac{p\lambda_2 s}{(1-\rho)}}+$$
$$(s_{12}+\frac{s_{21}}{1-\rho_1})\frac{1}{\frac{p\lambda_2 s}{(1-\rho)}}+\frac{1}{\lambda_2}\left[\frac{d}{dz}H_2(z)\right]_{z=1}\frac{1}{\frac{p\lambda_2 s}{(1-\rho)}}-\frac{1-(1-p)\rho_2}{\lambda_2 p},$$
$$p\in[0,1], \tag{3.2.4}$$

$$\text{with } H_2(z):=\prod_{k=1}^{\infty}\{D_p(\delta_p^{(k)}(z))/D_p(\delta_p^{(k)}(1))\}. \tag{3.2.5}$$

Remark 3.1.

By applying the chain rule to $D_p(\delta_p^{(k-1)}(\mu_2(\mu_1(z))))$ and noting that $\mu_1(1)=1$ we get the following relation between the infinite products in (3.1.5) and (3.2.5):

$$\left[\frac{d}{dz} H_2(z)\right]_{z=1} = \frac{\lambda_2\ \beta_1}{1-\rho_1} \left[\frac{d}{dz} H_1(\mu_2(z))\right]_{z=1}. \tag{3.2.6}$$

■

Remark 3.2.

Recently, the following expression for the pseudo conservation law for this polling system has been derived (cf. Boxma [1989], Tedijanto [1989]):

$$\rho_1 E\{w_1\} + \rho_2 \left[1 - \frac{p\lambda_2 s}{1-\rho}\right] E\{w_2\} =$$

$$\rho \frac{\lambda_1\beta_1^{(2)} + \lambda_2\beta_2^{(2)}}{2(1-\rho)} + \rho \frac{s^{(2)}}{2s} + \frac{s}{1-\rho}\rho_1\rho_2 + \frac{s}{1-\rho}\rho_2^2 p \tag{3.2.7}$$

Using relation (3.2.6) we find after a tedious but straightforward calculation that the expressions for the mean waiting times at Q_1 (cf. (3.1.4)) and at Q_2 (cf. (3.2.4)) satisfy this pseudo conservation law.

■

We have calculated $E\{w_1\}$ and $E\{w_2\}$ using formula (3.1.4),(3.2.5) and replacing

$\left[\frac{d}{dz}H_2(z)\right]_{z=1}$ by the finite sum $\sum_{k=1}^{N}\{\frac{d}{dz} D_p(\delta_p^{(k)}(z))|_{z=1}\}/D_p(\delta_p^{(k)}(1))$.

Table I to VII, and other numerical experiments, suggest that

-N increases as the Bernoulli parameter p increases,

-N increases as the workload at the Bernoulli queue, and subsequently of the whole system, increases.

As stop criterium we used the difference between the sum of the first (N-1) terms and the first N terms. In most cases considered $N \leq 6$ is sufficient to get a difference of less than 10^{-6}

Model I

The arrival process at Q_1 is Poisson with intensity λ_1 equal to 2.0. The service time distribution at this station is Exponential with mean β_1 equal to 0.05. The service strategy is Exhaustive. The switchover time from Q_1 to Q_2, s_{12}, is deterministic and equals 0.045.

The arrival process at Q_2 is Poisson with intensity λ_2 equal to 2.5. The service time distribution is Erlang-3 with mean β_2 equal to 0.09. The service strategy at this queue is Bernoulli with parameter p and the switchover time from Q_2 to Q_1, s_{21}, is deterministic and equal to 0.045.

In Model II to Model VII we change one or more parameters of Model I. InModel II we change both switchover times to 0.005. In model III we change the average service time at Q_2 to 0.30. In model IV we change both switchover times to 0.005 (as in Model II) and the mean service time at Q_2 to 0.30 (as in Model III). In Model V to VII we investigate the influence of the arrival intensity at the Bernoulli queue. In Model V we take $\lambda_2=2.0$, in Model VI $\lambda_2=1.5$ and in Model VII $\lambda_2=1.0$. In all those models $\beta_2=0.30$ (as in Model III).

In the tables we use the following shorthand notation:

$$\Sigma := \sum_{k=1}^{N} \{\frac{d}{dz} D_p(\delta_p^{(k)}(z))|_{z=1}\}/D_p(\delta_p^{(k)}(1)),$$

ergod:$=\rho + p\lambda_2 s$, cf. Remark 2.2.

§ 4 Summary and future work.

This paper has been devoted to the customers' waiting times in a polling system with two queues in which one queue has a Bernoulli(p) service strategy and the other queue an exhaustive service strategy. For this system we have derived exact expressions for the LST of the waiting time distributions via an iteration procedure. Based on those relations we expressed the customers' mean waiting times at both queues in the system parameters.

In a future study we would like to investigate the possibility

(1) to derive a conservation law based approximation for the mean waiting time at a Bernoulli(p) queue in a cyclic service system with $N \geq 2$ queues, using the results of this study.

(2) to use this approximation for system optimization.

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Table I, model I

p	EW1	EW2	$\sum$	N	ergod
0.1	0.0894	0.0845	0.0198	4	0.348
0.2	0.0877	0.0916	0.0188	4	0.370
0.3	0.0861	0.0991	0.0179	4	0.393
0.4	0.0845	0.1072	0.0170	4	0.415
0.5	0.0832	0.1158	0.0162	4	0.438
0.6	0.0817	0.1251	0.0153	3	0.460
0.7	0.0805	0.1352	0.0146	3	0.483
0.8	0.0793	0.1462	0.0139	3	0.505
0.9	0.0782	0.1582	0.0132	3	0.528
1.0	0.0771	0.1713	0.0125	2	0.550

Table II, modelII

EW1	EW2	$\sum$	N	ergod
0.0364	0.0331	0.0022	4	0.328
0.0350	0.0916	0.0021	3	0.330
0.0337	0.0353	0.0020	3	0.333
0.0325	0.0363	0.0019	3	0.335
0.0314	0.0372	0.0018	3	0.338
0.0304	0.0382	0.0017	3	0.340
0.0295	0.0391	0.0016	3	0.343
0.0286	0.0340	0.0015	3	0.345
0.0728	0.0408	0.0015	3	0.348
0.0271	0.0417	0.0014	2	0.350

Table III, model III

p	EW1	EW2	$\sum$	N	ergod
0.1	1.4918	1.3273	0.0721	6	0.873
0.2	0.9822	1.7731	0.0514	5	0.895
0.3	0.7364	2.3980	0.0400	5	0.918
0.4	0.5891	3.4589	0.0328	4	0.940
0.5	0.4898	5.7672	0.0274	4	0.963
0.6	0.4178	14.9640	0.0233	4	0.985
0.7					
0.8					
0.9					
1.0					

Table IV, model IV

EW1	EW2	$\sum$	N	ergod
1.3007	1.0322	0.0078	5	0.853
0.8244	1.1209	0.0055	5	0.855
0.6002	1.1773	0.0043	4	0.858
0.4684	1.2225	0.0035	4	0.860
0.3810	1.2629	0.0029	4	0.863
0.3186	1.3011	0.0025	3	0.865
0.2717	1.3384	0.0021	3	0.868
0.2350	1.3755	0.0018	3	0.870
0.2056	1.4130	0.0016	3	0.873
0.1814	1.4511	0.0014	2	0.875

Table V, model V

p	EW1	EW2	Σ	N	ergod
0.1	0.7198	0.4964	0.0375	6	0.718
0.2	0.5657	0.5800	0.0304	5	0.736
0.3	0.4689	0.6640	0.0255	5	0.754
0.4	0.4014	0.7549	0.0218	4	0.772
0.5	0.3512	0.8573	0.0190	4	0.790
0.6	0.3121	0.9759	0.0166	4	0.808
0.7	0.2806	1.1170	0.0146	4	0.826
0.8	0.2546	1.2888	0.0129	3	0.844
0.9	0.2327	1.5040	0.0113	3	0.862
1.0	0.2139	1.7824	0.0100	2	0.880

Table VI, model VI

EW1	EW2	Σ	N	ergod
0.3709	0.2676	0.0194	5	0.564
0.3261	0.2963	0.0170	5	0.577
0.2921	0.3243	0.0151	4	0.591
0.2651	0.3524	0.0135	4	0.604
0.2431	0.3812	0.0122	4	0.618
0.2247	0.4111	0.0110	4	0.631
0.2090	0.4425	0.0100	3	0.645
0.1955	0.4758	0.0091	3	0.658
0.1838	0.5113	0.0083	3	0.672
0.1731	0.5494	0.0075	2	0.658

Table VII, model VII

p	EW1	EW2	Σ	N	ergod
0.1	0.1975	0.1605	0.0092	4	0.409
0.2	0.1862	0.1715	0.0086	4	0.418
0.3	0.1764	0.1823	0.0080	4	0.427
0.4	0.1679	0.1931	0.0074	4	0.436
0.5	0.1603	0.2038	0.0069	4	0.445
0.6	0.1536	0.2146	0.0065	3	0.454
0.7	0.1475	0.2255	0.0061	3	0.463
0.8	0.1419	0.2365	0.0057	3	0.472
0.9	0.1369	0.2478	0.0053	3	0.481
1.0	0.1322	0.2593	0.0050	2	0.490

Appendix

In this appendix we shall prove that for $|z| \le 1$ and $p \in [0,1]$:

$$0 < \prod_{k=1}^{\infty} D_p(\delta_p^{(k)}(z)) < \infty . \qquad (A.1)$$

First we present some definitions, in which we suppress z for notational convenience, and a Lemma.

Define for k, n=1,2,.. and $|z| \le 1$:

$$B_2(k,n) := \tilde{B}_2\{\delta_p^{(k)}(z), \delta_p^{(n)}(z)\}; \qquad (A.2)$$

$$S_{ij}(k,n) := \tilde{S}_{ij}\{\delta_p^{(k)}(z), \delta_p^{(n)}(z)\}, \quad i,j \in \{1,2\}; \qquad (A.3)$$

$$D_p(k) := D_p(\delta_p^{(k)}(z)) \qquad (A.4)$$

Lemma A.1

$$\lim_{k \to \infty} \frac{1-D_p(k+1)}{1-D_p(k)} < 1.$$

Proof:

In this proof we suppose that $p \ne 0$. For the p=0 part of the proof see Eisenberg [1972].

$$\frac{1-D_p(k+1)}{1-D_p(k)} = F_1(k)F_2(k), \quad k=1,2,... \qquad (A.5)$$

with

$$F_1(k) := \frac{(1-p)[B_2(k-1,k)-B_2(k,k)] - pS_{12}S_{21}B_2(k,k)}{(1-p)[B_2(k,k+1)-B_2(k+1,k+1)] - pS_{12}S_{21}B_2(k+1,k+1)}, \quad \text{(A.6)}$$

$F_2(k) :=$

$$\frac{(1-p)[B_2(k,k+1)-B_2(k+1,k+1)][1-S_{12}(k,k+1)S_{21}(k+1,k+1)]}{(1-p)[B_2(k-1,k)-B_2(k,k)][1-S_{12}(k-1,k)S_{21}(k,k)]+p[S_{12}(k-1,k)-S_{12}(k,k)]S_{12}B_2(k,k)]}$$

$$+$$

$$\frac{p[S_{12}(k,k+1)-S_{12}(k+1,k+1)]S_{12}B_2(k+1,k+1)}{(1-p)[B_2(k-1,k)-B_2(k,k)][1-S_{12}(k-1,k)S_{21}(k,k)]+p[S_{12}(k-1,k)-S_{12}(k,k)]S_{12}B_2(k,k)]}.$$

(A.7)

From (A.6) it follows that:

$$\lim_{k\to\infty} F_1(k) = 1. \quad \text{(A.8)}$$

If we take the limit of $F_2(k)$ for k to infinity both the nominator and the denominator of $F_2(k)$ (cf. (A.7)) tend to zero. Applying L'Hôpistal's rule yields after a lengthy calculation:

$$\lim_{k\to\infty} F_2(k) = \frac{\lim_{k\to\infty} \mu_1'(\delta_p^{(k)}(z)) - \mu_1'(\delta_p^{(k+1)}(z))}{\lim_{k\to\infty} \mu_1'(\delta_p^{(k-1)}(z)) - \mu_1'(\delta_p^{(k)}(z))}, \quad \text{(A.9)}$$

with $\mu_1'(\delta_p^{(k)}(z)) := \frac{d}{dz}\mu_1(\delta_p^{(k)}(z))$.

Using the chain rule several times we can write for $k=1,2,..$ and $|z|\le 1$:

$$\frac{d}{dz}\mu_1(\delta_p^{(k)}(z))=\left[\frac{d}{dy}\mu_1(y)\right]_{y=\delta_p^{(k)}(z)}\frac{d}{dz}\delta_p^{(k)}(z)=$$

$$\left[\frac{d}{dy}\mu_1(y)\right]_{y=\delta_p^{(k)}(z)}\prod_{n=0}^{k-1}\left[\frac{d}{dy}\delta_p(y)\right]_{y=\delta_p^{(n)}(z)}. \qquad (A.10)$$

Combining (A.9) and (A.10) we get:

$$\lim_{k\to\infty}F_2(k)=\left[\frac{d}{dy}\delta_p(y)\right]_{y=a}<1 \qquad (A.11)$$

where the last inequality follows from Remark A.1 below.

It is now easy to prove that for $0\le z\le 1$:

$$\lim_{k\to\infty}\delta_p^{(k)}(z)=a \text{ for some } a\in(0,1].$$

■

Finally, combining (A.5), (A.8) and (A.11) proves our lemma:

$$\lim_{k\to\infty}\frac{1-D_p(k+1)}{1-D_p(k)}=\left[\frac{d}{dy}\delta_p(y)\right]_{y=a}<1.$$

□

<u>Remark</u> <u>A.1</u>

Note that:

1) all the derivatives of $\delta_p(z)$ with respect to z are positive on the

interval [0,1] for each $p\in(0,1]$, i.e.

$$\frac{d^k}{dz^k}\,\delta_p(z) > 0 \ , \ k=0,1,\ldots \ ,0\le z\le 1, \ p\in(0,1];$$

2) $\delta_p(0) > 0, \ p\in(0,1];$

3) $\delta_p(1) \le 1, \ p\in(0,1],$

It is now easy to prove that for $0\le z\le 1$:

$$\lim_{k\to\infty} \delta_p^{(k)}(z) = a \text{ for some } a\in(0,1].$$

■

<u>Theorem</u> <u>A.1</u>

$$0 < \prod_{k=1}^{\infty} D_p(k) < \infty$$

Proof:

From calculus (cf. Titchmarsh [1939]) we know that:

1) $0 < \prod_{k=1}^{\infty} D_p(k) < \infty \Leftrightarrow \sum_{k=1}^{\infty} [1-D_p(k)] < \infty \ ;$ (A.12)

2) $\lim_{k\to\infty} \dfrac{1-D_p(k+1)}{1-D_p(k)} < 1 \ \Rightarrow \sum_{k=1}^{\infty} [1-D_p(k)] < \infty \ .$ (A.13)

The theorem now follows from Lemma A.1.

□

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